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AN ARTIFICIAL BOUNDARY CONDITION FOR TWO-DIMENSIONAL INCOMPRESSIBLE VISCOUS FLOWS USING THE METHOD OF LINES

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SUMMARY

We design an artificial boundary condition for the steady incompressible Navier–Stokes equations in streamfunction–vorticity formulation in a flat channel with slip boundary conditions on the wall. The new boundary condition is derived from the Oseen equations and the method of lines. A numerical experiment for the non-linear Navier–Stokes equations is presented. The artificial boundary condition is compared with Dirichlet and Neumann boundary conditions for the flow past a rectangular cylinder in a flat channel. The numerical results show that our boundary condition is more accurate.

KEY WORDS: Navier–Stokes equations; Oseen equations; method of lines; artificial boundary condition

1. INTRODUCTION

Many numerical simulations of viscous flow problems in physically unbounded domains are carried out in 'truncated' bounded computational domains with an artificial boundary. Artificial boundary conditions such as Neumann or Dirichlet boundary conditions are then prescribed at the artificial boundary. In general the above artificial boundary conditions are only very rough approximations of the exact boundary condition at the artificial boundary. Hence the bounded computational domain must be quite large when high accuracy is required, so the cost of the computation is increased. In order to limit the computational cost, the artificial boundary is often chosen not too far from the domain of interest. The proper specification of the boundary condition at a given artificial boundary for solving partial differential equations on an unbounded domain has been studied. For example, Goldstein[1] and Feng[2] studied Helmholtz-type equations and designed asymptotic radiation conditions at the given circle artificial boundary. Han and Wu[3,4] presented a sequence of artificial boundary conditions with high accuracy for the Laplace equation and the linear elasticity system. Hagstrom and Keller[5] obtained the exact boundary condition and artificial boundary conditions at an artificial boundary for partial differential equations in a cylinder, which were used to solve the non-linear problem.[6] Halpern[7] and Halpern and Schatzman[8] developed a family of artificial boundary conditions for the unsteady Oseen equations, which was then applied to the unsteady Navier–Stokes (N–S) equations. Nataf[9] designed an open boundary condition for the steady Oseen equations in a flat channel with slip boundary conditions on the wall. Hagstrom[10,11] proposed asymptotic boundary conditions at an artificial boundary for the simulation of time-dependent fluid flows. Recently Han *et al.*[12] designed a discrete artificial boundary condition for a system of linear N–S equations in a flat channel with no-slip boundary conditions on the wall.

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The purpose of this paper is to design a discrete artificial boundary condition for steady incompressible viscous flows in a channel using the method of lines.[13]

2. NAVIER–STOKES EQUATIONS AND OSEEN EQUATIONS

Throughout this paper we consider the numerical simulation of a steady incompressible viscous flow surrounding a body (domain Ω_i) in a channel defined by $\mathbb{R} \times [0, L]$ with a slip boundary condition on the wall. The N–S equations in the domain $\Omega = \mathbb{R} \times (0, L) \setminus \overline{\Omega}_i$ are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \nu\Delta u, \tag{1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \nu\Delta v, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

and the boundary conditions are

$$v|_{y=0,L} = 0 \quad \text{and} \quad \sigma_{12}|_{y=0,L} = \nu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\bigg|_{y=0,L} = 0, \quad -\infty < x < \infty, \tag{4}$$

$$u|_{\partial\Omega_i} = v|_{\partial\Omega_i} = 0, \tag{5}$$

$$u(x, y) \to u_\infty = \text{const.} \quad \text{and} \quad v(x, y) \to 0 \quad \text{when } x \to \pm\infty, \tag{6}$$

where u and v are the components of velocity in the x- and y-co-ordinate directions respectively, p is the pressure, $\nu > 0$ is the kinematic viscosity and σ_{12} is the tangential stress on the wall.

Let ψ and ω denote the streamfunction and vorticity; then

$$\frac{\partial \psi}{\partial y} = u, \qquad \frac{\partial \psi}{\partial x} = -v, \tag{7}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8}$$

From the equations and boundary conditions (1)–(6) we have

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} - \nu\Delta\omega = 0 \quad \text{in } \Omega, \tag{9}$$

$$\Delta\psi + \omega = 0 \quad \text{in } \Omega, \tag{10}$$

$$\psi|_{y=0} = \omega|_{y=0,L} = 0 \quad \text{and} \quad \psi|_{y=L} = \psi_L \equiv u_\infty L, \quad -\infty < x < \infty, \tag{11}$$

$$\psi|_{\partial\Omega_i} = \text{const.}, \qquad \left|\frac{\partial \psi}{\partial n}\right|_{\partial\Omega_i} = 0, \tag{12}$$

$$\psi \to \psi_\infty(y) \equiv u_\infty y \quad \text{and} \quad \omega \to 0 \quad \text{when } x \to \pm\infty, \tag{13}$$

where $\partial/\partial n$ denotes the outward normal derivative.

Taking two constants $b < c$ such that $\overline{\Omega}_i \subset (b, c) \times (0, L)$, Ω is divided into three parts Ω^b, Ω^T and Ω^c by the artificial boundaries $\Gamma_b = \{x = b,\ 0 \leqslant y \leqslant L\}$ and $\Gamma_c = \{x = c,\ 0 \leqslant y \leqslant L\}$, with

$$\Omega^b = \{(x, y) | -\infty < x < b,\ 0 < y < L\},$$
$$\Omega^T = \{(x, y) | b < x < c,\ 0 < y < L\} \setminus \overline{\Omega}_i,$$
$$\Omega^c = \{(x, y) | c < x < \infty,\ 0 < y < L\}.$$

When $|b|$ and c are sufficiently large, in the domain $\Omega^b \cup \Omega^c$ the velocity (u, v) is an almost constant vector $(u_\infty, 0)$. Thus the N–S equations (1)–(3) can be linearized in the domain Ω^c (and Ω^b), namely the solution ω, and ψ of the problem (7)–(13) approximately satisfies the problem

$$\Delta\omega - u_\infty\, Re \frac{\partial \omega}{\partial x} = 0 \quad \text{in } \Omega^c, \tag{14}$$

$$\Delta\psi + \omega = 0 \quad \text{in } \Omega^c, \tag{15}$$

$$\psi|_{y=0} = \omega|_{y=0,\, L} = 0 \quad \text{and} \quad \psi|_{y=L} = \psi_L, \quad c \leqslant x < \infty, \tag{16}$$

$$\psi \to \psi_\infty(y) \quad \text{and} \quad \omega \to 0 \quad \text{when } x \to \infty, \tag{17}$$

where $Re = 1/\nu$. Let

$$\tilde{\omega}(x, y) = \omega(x, y), \qquad \tilde{\psi}(x, y) = \psi(x, y) - \psi_\infty(y). \tag{18}$$

Since $\psi_\infty(y)$ is a polynomial of degree one, it is straightforward to check that $\tilde{\omega}$ and $\tilde{\psi}$ satisfy equations (14) and (15) and the boundary conditions

$$\tilde{\psi}|_{y=0,\, L} = \tilde{\omega}|_{y=0,\, L} = 0, \quad c \leqslant x < \infty, \tag{19}$$

$$\tilde{\psi}(x, y) \to 0 \quad \text{and} \quad \tilde{\omega}(x, y) \to 0 \quad \text{when } x \to \infty. \tag{20}$$

Because the boundary condition on the artificial boundary Γ_c is unknown, equations (14) and (15) with boundary conditions (19) and (20) represent an incompletely posed problem. It cannot be solved. Let

$$\tilde{\psi}|_{x=c} = \tilde{\psi}_c(y) \quad \text{and} \quad \tilde{\omega}|_{x=c} = \tilde{\omega}_c(y), \quad 0 \leqslant y \leqslant L. \tag{21}$$

For given functions $\tilde{\psi}_c(y)$ with $\tilde{\psi}_c(0) = \tilde{\psi}_c(L) = 0$ and $\tilde{\omega}_c(y)$ with $\tilde{\omega}_c(0) = \tilde{\omega}_c(L) = 0$ we discuss the numerical solution of equations (14) and (15) with boundary conditions (19)–(21) and design a discrete artificial boundary condition on the segment Γ_c for the problem (7)–(13).

3. AN ARTIFICIAL BOUNDARY CONDITION

We now consider the semidiscretization approximation of the problem (14), (15), (19)–(21). Let $\Delta y = L/N$ be the mesh size, where N is a positive integer. The domain Ω^c is divided into N strips, i.e. $\overline{\Omega^c} = \cup_{k=1}^{N} \overline{\Omega}_k$, where

$$\Omega_k = \{(x, y) | c < x < \infty,\ (k-1)\Delta y = y_{k-1} < y < y_k = k\Delta y\}.$$

The following semidiscretization scheme is used to solve equations (14) and (15) with boundary conditions (19)–(21):

$$\frac{d^2\tilde{\omega}_k(x)}{dx^2}+\frac{\tilde{\omega}_{k+1}(x)-2\tilde{\omega}_k(x)+\tilde{\omega}_{k-1}(x)}{\Delta y^2}-u_\infty\,Re\frac{d\tilde{\omega}(x)}{dx}=0, \tag{22}$$

$$\frac{d^2\tilde{\psi}_k(x)}{dx^2}+\frac{\tilde{\psi}_{k+1}(x)-2\tilde{\psi}_k(x)+\tilde{\psi}_{k-1}(x)}{\Delta y^2}+\tilde{\omega}_k(x)=0, \tag{23}$$

$$1\leqslant k\leqslant N-1,$$

with boundary conditions

$$\tilde{\psi}_0(x)=\tilde{\psi}_N(x)=\tilde{\omega}_0(x)=\tilde{\omega}_N(x)=0,\quad c\leqslant x<\infty, \tag{24}$$

$$\tilde{\psi}_k(c)=\tilde{\psi}_c(y_k)\quad\text{and}\quad\tilde{\omega}_k(c)=\tilde{\omega}_c(y_k),\quad 1\leqslant k\leqslant N-1, \tag{25}$$

$$\lim_{x\to\infty}\tilde{\psi}_k(x)=\lim_{x\to\infty}\tilde{\omega}_k(x)=0,\quad 1\leqslant k\leqslant N-1. \tag{26}$$

Let

$$X_0=[\tilde{\omega}_c(y_1),\ \ldots,\ \tilde{\omega}_c(y_{N-1}),\ \tilde{\psi}_c(y_1),\ \ldots,\ \tilde{\psi}_c(y_{N-1})]^{\mathrm{T}},$$

$$X(x)=[\tilde{\omega}_1(x),\ \ldots,\ \tilde{\omega}_{N-1}(x),\ \tilde{\psi}_1(x),\ \ldots,\ \tilde{\psi}_{N-1}(x)]^{\mathrm{T}}.$$

Then the problem (22)–(26) is equivalent to the following ordinary differential system with constant coefficients. Find X such that

$$\ddot{X}(x)+A_0\dot{X}(x)+B_0X(x)=0, \tag{27}$$

$$X(c)=X_0,\qquad \lim_{x\to\infty}X(x)=0, \tag{28}$$

where A_0 and B_0 are $2(N-1)\times 2(N-1)$ matrices given by

$$A_0=-u_\infty\,Re\begin{pmatrix} I_{N-1} & 0\\ 0 & 0\end{pmatrix},\qquad B_0=\begin{pmatrix} D_0 & 0\\ I_{N-1} & D_0\end{pmatrix};$$

here I_{N-1} is the $(N-1)\times(N-1)$ unit matrix and

$$D_0=\beta\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0\\ 1 & -2 & 1 & \cdots & 0 & 0\\ 0 & 1 & -2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & -2 & 1\\ 0 & 0 & \cdots & 0 & 1 & -2\end{pmatrix},$$

with

$$\beta=\frac{1}{\Delta y^2}.$$

Because A_0 and B_0 are constant matrices, we can get the solution of the problem (27), (28) directly. Let $e^{\lambda(x-c)}\varsigma$ be a solution of the system of ordinary differential equations in (27). Then we know that constant λ and non-zero vector ς are a solution of the eigenvalue problem

$$(\lambda^2 I_{2N-2} + \lambda A_0 + B_0)\,\varsigma = 0. \tag{29}$$

Using the special construction of the matrices A_0 and B_0, we can get the eigenvalues and eigenvectors of (29) immediately. In fact, since $\varsigma \neq 0$, we have

$$\begin{aligned} 0 &= \det(\lambda^2 I_{2N-2} + \lambda A_0 + B_0) \\ &= \det\begin{pmatrix} (\lambda^2 - u_\infty\, Re\, \lambda) I_{N-1} + D_0 & 0 \\ I_{N-1} & \lambda^2 I_{N-1} + D_0 \end{pmatrix} \\ &= \det((\lambda^2 - u_\infty\, Re\, \lambda) I_{N-1} + D_0]\det(\lambda^2 I_{N-1} + D_0), \end{aligned}$$

namely

$$\det(\lambda^2 I_{N-1} + D_0) = 0 \quad \text{or} \quad \det[(\lambda^2 - \lambda u_\infty\, Re) I_{N-1} + D_0] = 0. \tag{30}$$

Since D_0 is a special tridiagonal $N-1$ matrix, the eigenvalues $\{\mu_i\}$ and corresponding eigenvectors $\{\eta_i\}$ of D_0 are given by

$$\mu_i = 2\beta\left[-1 + \cos\left(\frac{i\pi}{N}\right)\right], \quad 1 \leqslant i \leqslant N-1,$$

$$\eta_i = \frac{1}{\sqrt{\{\sum_{l=1}^{N-1}[\sin(il\pi/N)]^2\}}}\left[\sin\left(\frac{i\pi}{N}\right), \sin\left(\frac{2i\pi}{N}\right), \ldots, \sin\left(\frac{(N-1)i\pi}{N}\right)\right]^{\mathrm{T}}. \tag{31}$$

Hence we obtain the eigenvalues of the problem (29) as

$$\lambda_i^2 = 2\beta\left[1 - \cos\left(\frac{i\pi}{N}\right)\right], \quad 1 \leqslant i \leqslant N-1$$

$$\left(\lambda_i - \frac{Re\, u_\infty}{2}\right)^2 = \frac{Re^2\, u_\infty^2}{4} + 2\beta\left[1 - \cos\left(\frac{(i+1-N)\pi}{N}\right)\right], \quad N \leqslant i \leqslant 2N-2. \tag{32}$$

From the boundary condition $\lim_{x\to\infty} X(x) = 0$, λ_i must be taken negative; then we have

$$\lambda_i = \begin{cases} -\sqrt{\left\{2\beta\left[1 - \cos\left(\frac{i\pi}{N}\right)\right]\right\}}, & 1 \leqslant i \leqslant N-1, \\ \frac{Re\, u_\infty}{2} - \sqrt{\left\{\frac{Re^2\, u_\infty^2}{4} + 2\beta\left[1 - \cos\left(\frac{(i+1-N)\pi}{N}\right)\right]\right\}} & N \leqslant i \leqslant 2N-2. \end{cases}$$

Let

$$a_i = -Re\, u_\infty \lambda_i, \quad N \leqslant i \leqslant 2N-2,$$

$$z_i = \eta_i, \quad 1 \leqslant i \leqslant N-1.$$

By computation we obtain the eigenvectors ς_i ($1 \leqslant i \leqslant 2N-2$) of the problem (29) corresponding to λ_i:

$$\varsigma_i = \begin{cases} [0, 0, \ldots, 0, z_i^{\mathrm{T}}]^{\mathrm{T}}, & 1 \leqslant i \leqslant N-1, \\ \dfrac{a_i}{\sqrt{(1+a_i^2)}}\left[z_{i+1-N}^{\mathrm{T}}, \dfrac{1}{a_i} z_{i+1-N}^{\mathrm{T}}\right]^{\mathrm{T}}, & N \leqslant i \leqslant 2N-2. \end{cases}$$

Thus

$$X(x) = \sum_{j=1}^{2N-2} b_j e^{\lambda_j(x-c)} \varsigma_j, \tag{33}$$

satisfied the system of ordinary differential equations (27) and the boundary condition $\lim_{x\to\infty} X(x) = 0$ for any constants $b_1, b_2, \ldots, b_{2N-2}$.

Differentiating (33), we have

$$\dot{X}(x) = \sum_{j=1}^{2N-2} \lambda_j b_j e^{\lambda_j(x-c)} \varsigma_j. \tag{34}$$

Introducing the matrices Y and $\tilde{Y}$ and the vector b as

$$Y = [\varsigma_1, \varsigma_2, \ldots, \varsigma_{2N-2}], \qquad \tilde{Y} = [\lambda_1\varsigma_1, \lambda_2\varsigma_2, \ldots, \lambda_{2N-2}\varsigma_{2N-2}], \qquad b = [b_1, b_2, \ldots, b_{2N-2}]^{\mathrm{T}},$$

from (33) and (34) we obtain

$$X(c) = Y b, \tag{35}$$

$$\dot{X}(c) = \tilde{Y} b, \tag{36}$$

$$\dot{X}(c) = \tilde{Y} Y^{-1} X(c). \tag{37}$$

Let

$$Z(x) = [\omega_1(x), \omega_2(x), \ldots, \omega_{N-1}(x), \psi_1(x), \psi_2(x), \ldots, \psi_{N-1}(x)]^{\mathrm{T}};$$

then

$$Z(c) = X(c) + d, \qquad \dot{Z}(c) = \dot{X}(c), \tag{38}$$

where

$$d = [0, 0, \ldots, 0, u_\infty \Delta y, 2u_\infty \Delta y, \ldots, (N-1)u_\infty \Delta y]^{\mathrm{T}} \in \mathbb{R}^{2N-2}.$$

Substituting (37) into (38), we obtain the following discrete artificial boundary condition on the artificial boundary Γ_c:

$$\dot{Z}(c) = TZ(c) + S, \tag{39}$$

with

$$T = \tilde{Y} Y^{-1}, \qquad S = -\tilde{Y} Y^{-1} d.$$

In a similar way we can get the artificial boundary condition on the boundary Γ_b.

Assume that $\{l_k(y),\ k = 0, 1, \ldots, N\}$ is a basic set of interpolating functions of Γ_c, e.g.

$\{l_k(y),\ k = 0, 1, \ldots, N\}$ is Lagrange interpolating polynomial. Then we have

$$\frac{\partial \omega(c,y)}{\partial x} = \sum_{k=0}^{N} l_k(y)\frac{\partial \omega(c,y_k)}{\partial x} = \sum_{k=1}^{N-1} l_k(y)\frac{\partial \omega(c,y_k)}{\partial x}, \tag{40}$$

$$\frac{\partial \psi(c,y)}{\partial x} = \sum_{k=0}^{N} l_k(y)\frac{\partial \psi(c,y_k)}{\partial x} = \sum_{k=1}^{N-1} l_k(y)\frac{\partial \psi(c,y_k)}{\partial x}. \tag{41}$$

The last equalities in (40) and (41) are from

$$\frac{\partial \omega(c,y_0)}{\partial x} = \frac{\partial x \omega(c,y_N)}{\partial x} = 0, \qquad \frac{\partial \psi(c,y_0)}{\partial x} = \frac{\partial \psi(c,y_N)}{\partial x} = 0,$$

Then we have

$$\begin{aligned}\begin{pmatrix} \dfrac{\partial \omega(c,y)}{\partial x} \\ \dfrac{\partial \psi(c,y)}{\partial x} \end{pmatrix} &= \begin{pmatrix} \sum_{k=1}^{N-1} l_k(y)\dfrac{\partial \omega(c,y_k)}{\partial x} \\ \sum_{k=1}^{N-1} l_k(y)\dfrac{\partial \psi(c,y_k)}{\partial x} \end{pmatrix} \\ &= L_i\left(\frac{\partial \omega(c,y_1)}{\partial x}\cdots\frac{\partial \omega(c,y_{N-1})}{\partial x}\ \frac{\partial \psi(c,y_1)}{\partial x}\cdots\frac{\partial \psi(c,y_{N-1})}{\partial x}\right)^{\mathrm{T}} \\ &= L_i(TZ(c)+S) \\ &\equiv \Pi_{\mathrm{h}}^{c}(\omega,\psi),\end{aligned} \tag{42}$$

where

$$L_i = \begin{pmatrix} l_1(y) & \cdots & l_{N-1}(y) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & l_1(y) & \cdots & l_{N-1}(y) \end{pmatrix}.$$

Then on the domain Ω^{T} the original problem (7)–(13) can be approximated by equations (7) and (8) and

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} - \nu\Delta\omega = 0 \quad \text{in } \Omega^{\mathrm{T}}, \tag{43}$$

$$\Delta\psi + \omega = 0 \quad \text{in } \Omega^{\mathrm{T}}, \tag{44}$$

$$\psi|_{y=0} = \omega|_{y=0,L} = 0 \quad \text{and} \quad \psi|_{y=L} = \psi_L, \quad b \leqslant x \leqslant c, \tag{45}$$

$$\psi|_{\partial\Omega_i} = \text{const.}, \qquad \left|\frac{\partial \psi}{\partial n}\right|_{\partial\Omega_i} = 0, \tag{46}$$

$$\psi|_{\Gamma_b} = \psi_\infty(y), \qquad \omega|_{\Gamma_b} = 0, \tag{47}$$

$$\left.\left(\frac{\partial \omega}{\partial x}, \frac{\partial \psi}{\partial x}\right)^{\mathrm{T}}\right|_{\Gamma_c} = \Pi_{\mathrm{h}}^{c}(\omega,\psi). \tag{48}$$

Furthermore, we can also use a higher-order semidiscretization scheme, such as a fourth-order standard finite difference scheme or a Chebyshev scheme, to discretize equations (14) and (15) with boundary conditions (19)–(21). Then the problem is also reduced to the ordinary differential equation system with constant coefficients given by (27) and (28), which can be reduced to the eigenvalue problem (29), but the constructions of the matrices A_0 and B_0 are different for the various semidiscretization schemes. In the general case the solution of the eigenvalue problem (29) cannot be given directly. Let $\eta = \lambda\varsigma$; then the problem (29) is equivalent to the following standard eigenvalue problem. Find $\lambda \in \mathbb{R}$ and a non-zero vector $(\varsigma, \eta)^{\mathrm{T}} \in \mathbb{R}^{4N-4}$ such that

$$\begin{pmatrix} 0 & I \\ -B_0 & -A_0 \end{pmatrix} \begin{pmatrix} \varsigma \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \varsigma \\ \eta \end{pmatrix}. \tag{49}$$

The problem (49) can be solved by numerical methods and the approximate eigenvalues and corresponding eigenvectors of (49) can be obtained. Similarly we can obtain the discrete artificial boundary conditions by using different semidiscretization schemes.

4. NUMERICAL IMPLEMENTATION AND EXAMPLE

We now consider the numerical solution of the original problem (7)–(13) on the given computational domain Ω^T. The steady state solution is computed as the limit in time of the unsteady N–S equations, which are discretized by an ADI method.[14] The inflow boundary conditions

$$\psi(b, y) = \psi_\infty(y) \quad \text{and} \quad \omega(b, y) = 0, \quad 0 \leqslant y \leqslant L, \tag{50}$$

are prescribed on the artificial boundary Γ_b. On the artificial boundary Γ_c the following three different types of outflow boundary conditions on ω and ψ are used in the example for comparison.

Type I. Dirichlet boundary conditions

$$\psi(c, y) = \psi_\infty(y) \quad \text{and} \quad \omega(c, y) = 0, \quad 0 \leqslant y \leqslant L.$$

Type II. Neumann boundary conditions

$$\frac{\partial\psi}{\partial x}(c, y) = 0 \quad \text{and} \quad \frac{\partial\omega}{\partial x}(c, y) = 0, \quad 0 \leqslant y \leqslant L.$$

Type III. Discrete artificial boundary condition (39) or (42).

In the example the results are compared with an 'exact solution'. This solution is obtained by using an outflow boundary very far from the obstacle and with Nuemann boundary conditions on this outflow boundary. To be precise, the distance between the inflow boundary and the outflow boundary for the 'exact solution' is 11 times as long as H, where H is the length of the edge of the obstacle. In our example $H = 0\cdot4$.

Example

Consider the fluid flow in a horizontal channel with a rectangular cylinder obstacle. The obstacle is defined by the domain

$$\Omega_{\mathrm{i}} = \{(x, y) \mid 0\cdot8 < x < 1\cdot2,\ \tfrac{2}{5}L < y < \tfrac{3}{5}L\}.$$

The bounded computational domain Ω^T is given by

$$\Omega^T = \{(x, y) \mid b < x < c,\ 0 < y < L\} \setminus \bar{\Omega}_{\mathrm{i}}.$$

and $\psi_\infty(y) = u_\infty y$. We take $b = 0$, $L = 1\cdot0$, and $u_\infty = 1\cdot0$.

Table I. $Re = 10$, $c = d = 2{\cdot}4$

Error	$i = \text{I}$	$i = \text{II}$	$i = \text{III}$
$\text{err}(\omega_E - \omega_i)$	0·4167	0·1593	$2{\cdot}9279 \times 10^{-3}$
$\text{err}(\psi_E - \psi_i)$	$1{\cdot}3476 \times 10^{-2}$	$8{\cdot}8275 \times 10^{-3}$	$2{\cdot}2779 \times 10^{-4}$

Table II. $Re = 50$, $c = d = 2{\cdot}4$

Error	$i = \text{I}$	$i = \text{II}$	$i = \text{III}$
$\text{err}(\omega_E - \omega_i)$	3·9956	0·2499	$1{\cdot}2960 \times 10^{-2}$
$\text{err}(\psi_E - \psi_i)$	0·1040	$1{\cdot}7086 \times 10^{-2}$	$5{\cdot}0647 \times 10^{-4}$

Table III. $Re = 100$, $c = d = 2{\cdot}8$

Error	$i = \text{I}$	$i = \text{II}$	$i = \text{III}$
$\text{err}(\omega_E - \omega_i)$	4·7021	0·1473	$1{\cdot}5132 \times 10^{-2}$
$\text{err}(\psi_E - \psi_i)$	0·1202	$1{\cdot}0591 \times 10^{-2}$	$6{\cdot}3449 \times 10^{-4}$

Let (ω_E, ψ_E) denote the 'exact solution' and (ω_i, ψ_i) (i = I, II, III) denote the numerical solutions corresponding to the boundary conditions of types I, II and III respectively on the artificial boundary Γ_c. The errors $\omega_E - \omega_i$ and $\psi_E - \psi_i$ on the segment $\Gamma_d = \{x = d,\ 0 \leqslant y \leqslant L\}$ are given for various Reynolds numbers. Let

$$\text{err}(f_E - \bar{f}_i) = \sqrt{\left(\sum_{j=1}^{N} (f_E(d, y_j) - \bar{f}_i(d, y_j))^2\right)}.$$

Then the errors $\text{err}(\omega_E - \omega_i)$ and $\text{err}(\psi_E - \psi_i)$ are given in Tables I–III for $Re = 10$, 50 and 100 respectively.

Furthermore, the errors $\omega_E - \omega_i$ and $\psi_E - \psi_i$ on the segment Γ_d for $Re = 100$ are shown in Figures 1 and 2 respectively.

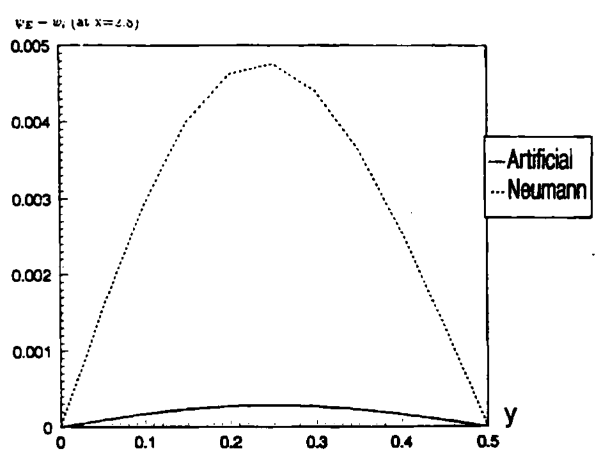

Figure 1. Error $\omega_E - \omega_i$ on segment Γ_d; $Re = 100$, $c = d = 2{\cdot}8$

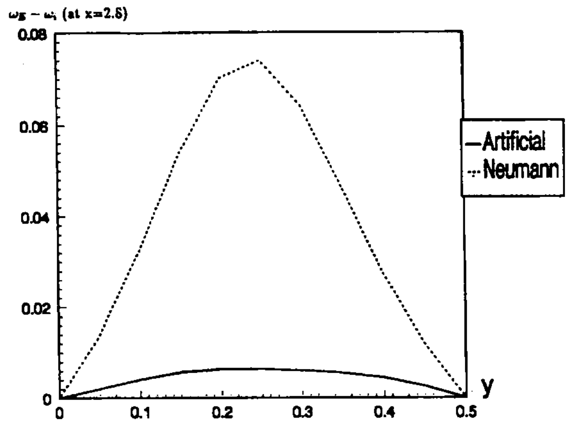

Figure 2. Error $\psi_E = \psi_i$ on segment Γ_d; $Re = 100$, $c = d = 2{\cdot}8$

Table IV. $Re = 10$, $d=2{\cdot}0$

	$c = 2{\cdot}0$	$2{\cdot}4$	$2{\cdot}8$	$3{\cdot}2$
$\mathrm{err}(\omega_E - \omega_{III}$	$1{\cdot}214 \times 10^{-2}$	$2{\cdot}518 \times 10^{-4}$	$7{\cdot}706 \times 10^{-5}$	$1{\cdot}264 \times 10^{-5}$
$\mathrm{err}(\psi_E - \psi_{III})$	$7{\cdot}378 \times 10^{-4}$	$1{\cdot}571 \times 10^{-5}$	$1{\cdot}724 \times 10^{-6}$	$2{\cdot}899 \times 10^{-7}$

Table V. $Re = 50$, $d = 2{\cdot}0$

	$c = 2{\cdot}0$	$2{\cdot}4$	$2{\cdot}8$	$3{\cdot}2$
$\mathrm{err}(\omega_E - \omega_{III})$	$4{\cdot}257 \times 10^{-2}$	$2{\cdot}626 \times 10^{-4}$	$2{\cdot}202 \times 10^{-5}$	$6{\cdot}023 \times 10^{-6}$
$\mathrm{err}(\psi_E - \psi_{III})$	$1{\cdot}508 \times 10^{-3}$	$4{\cdot}300 \times 10^{-5}$	$3{\cdot}410 \times 10^{-7}$	$2{\cdot}456 \times 10^{-7}$

Table VI. $Re = 100$, $d = 2{\cdot}4$

	$c = 2{\cdot}4$	$2{\cdot}8$	$3{\cdot}2$	$3{\cdot}6$
$\mathrm{err}(\omega_E - \omega_{III})$	$3{\cdot}384 \times 10^{-2}$	$3{\cdot}574 \times 10^{-4}$	$6{\cdot}226 \times 10^{-5}$	$1{\cdot}959 \times 10^{-5}$
$\mathrm{err}(\psi_E - \psi_{III})$	$1{\cdot}339 \times 10^{-3}$	$5{\cdot}342 \times 10^{-5}$	$1{\cdot}605 \times 10^{-7}$	$6{\cdot}024 \times 10^{-7}$

Tables I-III and Figures 1 and 2 show that the artificial boundary condition presented in this paper is more accurate than the Neumann and Dirichlet boundary conditions which are often used in the engineering literature.

The influence of the artificial boundary location Γ_c is shown in Tables IV–VI for various Reynolds numbers.

The location of the artificial boundary has a strong influence on the computational accuracy.

5. CONCLUSIONS

An artificial boundary condition for the Navier–Stokes equations has been formulated based on an external linear flow field and the method of lines. Even though the artificial boundary condition is obtained from the Oseen equations on an external domain, it can be used to solve the non-linear Navier–Stokes equations. From the numerical results we can see that our artificial boundary condition is more accurate than the Neuman and Dirichlet boundary conditions which are often used in engineering. For a given accuracy it is possible to compute the problem on a smaller computational domain by using our artificial boundary condition, thus saving computing time.

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